ASTR 400/700: Stellar Astrophysics

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Stellar Interiors

Chapter 10.1, 10.2, 10.3

Stellar Structure

Stellar evolution results from a constant battle with gravity!

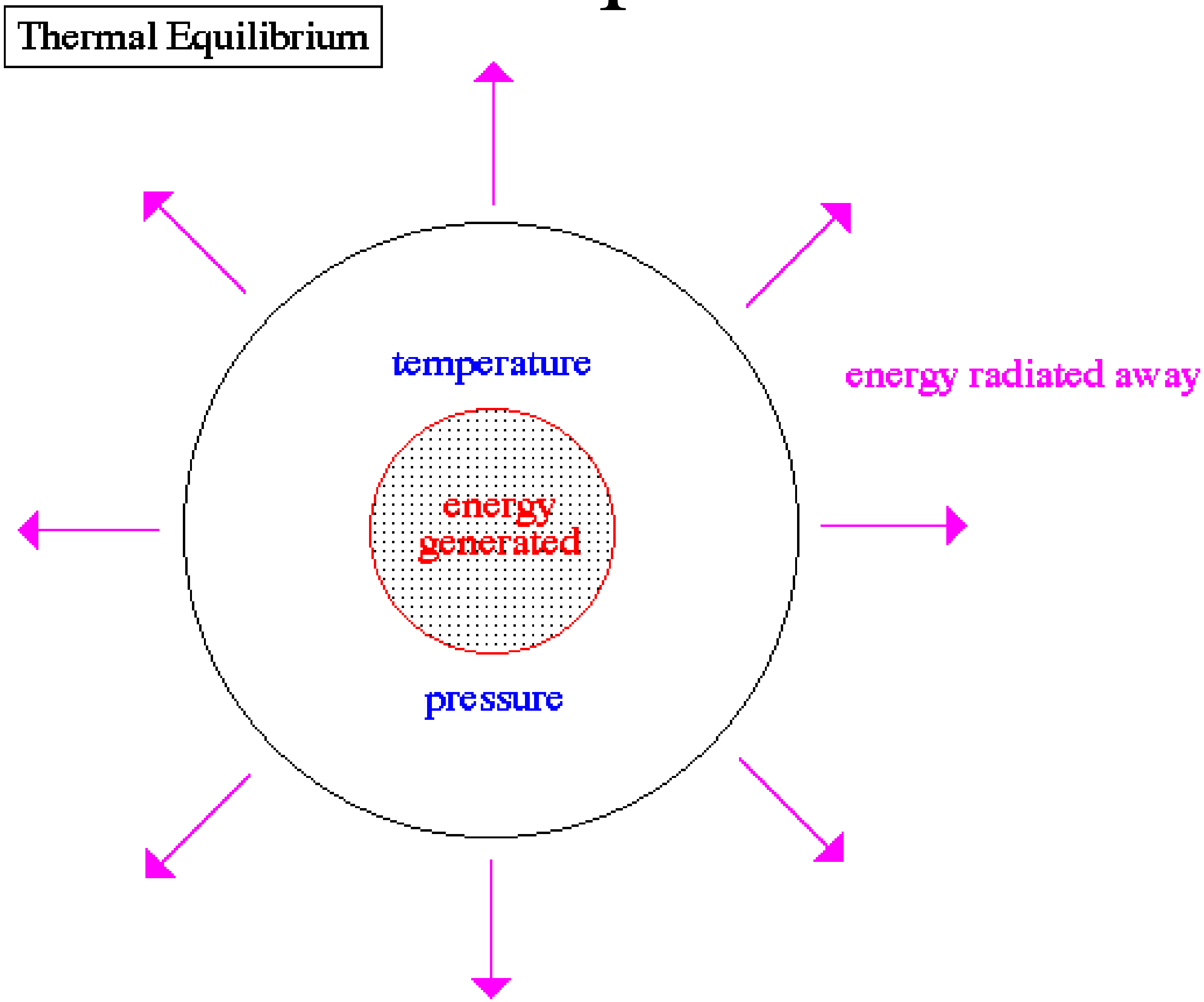
- Variation of temperature
- Variation of density
- Variation of pressure

Test against observations with computer models.

Physical Principles

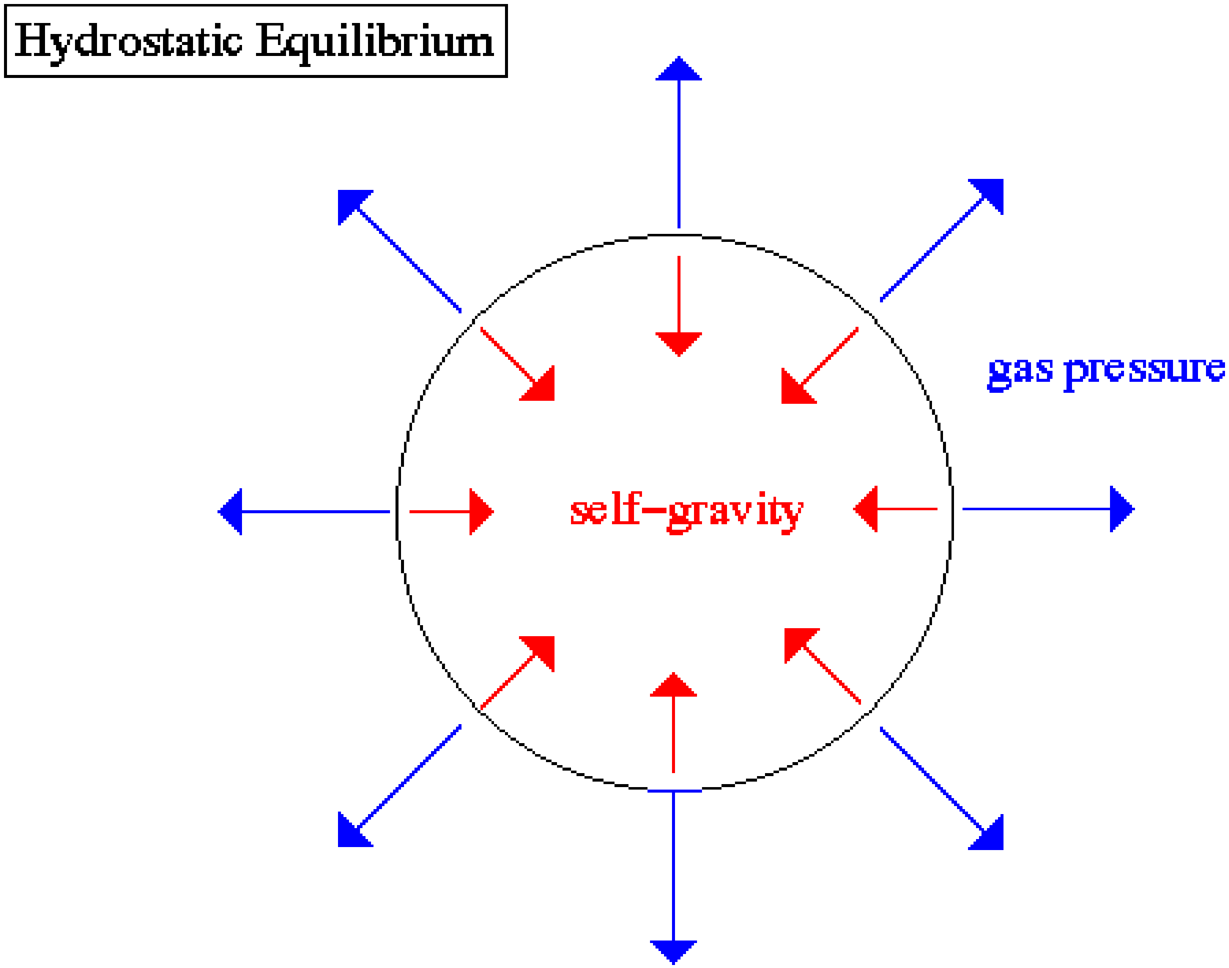
- Hydrostatic Equilibrium
- Virial Theorem
- Energy Generation
- Energy Transport

Thermal Equilibrium



the energy generated must be radiated away, if the energy production is increased, the temperature goes up, therefore the pressure goes up and the star expands – the surface area increases and more energy is radiated to space to balance the increased production

Hydrostatic Equilibrium



the Sun is not expanding or contracting, therefore it is in equilibrium,
the downward force of gravity is balanced by the higher force of pressure

Models of Stars

The parameters used for studying and modeling stellar interiors include:

r	=	**radial distance from the center of the star**
$M(r)$	=	**mass interior to** r
$T(r)$	=	**temperature at** r
$P(r)$	=	**pressure at** r
$L(r)$	=	**luminosity at** r
$\varepsilon(r)$	=	**energy generation at** r
$\kappa(r)$	=	**opacity at** r
$\rho(r)$	=	**density at** r

In modern models mass M **is used as the dependent variable rather than radial distance** r**, but it is more informative to initiate the study of stellar interiors using the geometrical variable** r**.**

At the “natural” boundaries of the star the corresponding values are:

- **At the center: At the surface:**

At the center			At the surface		
r	=	0	r	=	R
$M(r)$	=	0	$M(r)$	=	M_*
$T(r)$	=	T_c	$T(r)$	=	0 (or T_{eff})
$P(r)$	=	P_c	$P(r)$	=	0
$L(r)$	=	L_c	$L(r)$	=	L_*
$\rho(r)$	=	ρ_c	$\rho(r)$	=	0

Rotation and magnetic fields are usually ignored in most models (*i.e.* spherical symmetry is imposed), as well as any temporal changes (*i.e.* radial pulsation).

Hydrostatic Equilibrium

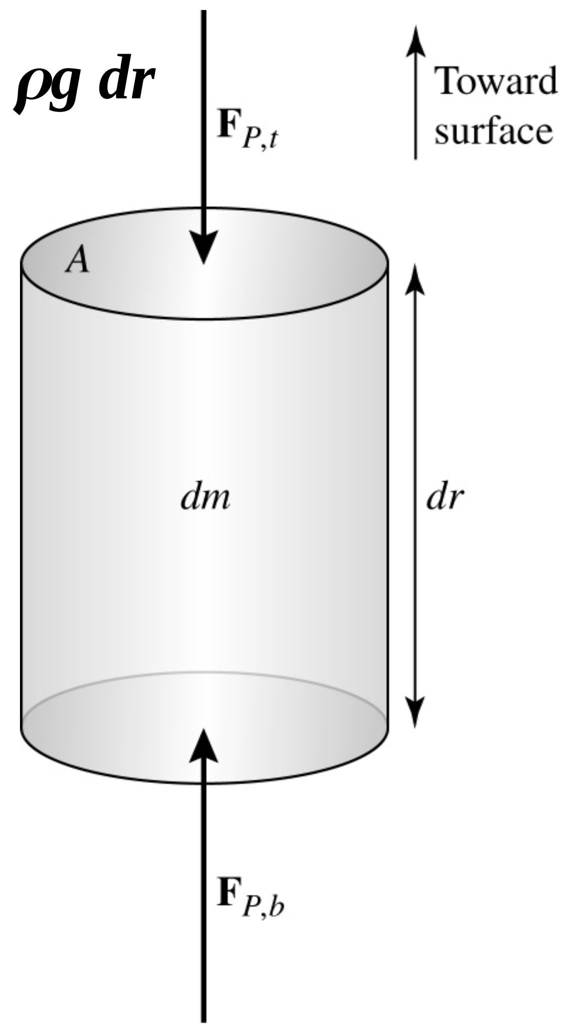
For balance at any point in the interior of a star, the weight of a cylinder of matter of unit cross-sectional area and thickness *dr* must be balanced by the buoyancy force of the gas pressure, *i.e.*:

Mass of cylinder = density × volume = $\rho A dr$
Weight of the cylinder = mass × local gravity = $\rho g\, dr$
But local gravity, $g = GM(r)/r^2$
Buoyant Force = pressure difference (top – bottom) = –dP
So:

$$-dP = \frac{G M(r) \rho\, dr}{r^2}$$

or

$$\frac{dP}{dr} = \frac{-G M(r) \rho}{r^2}$$



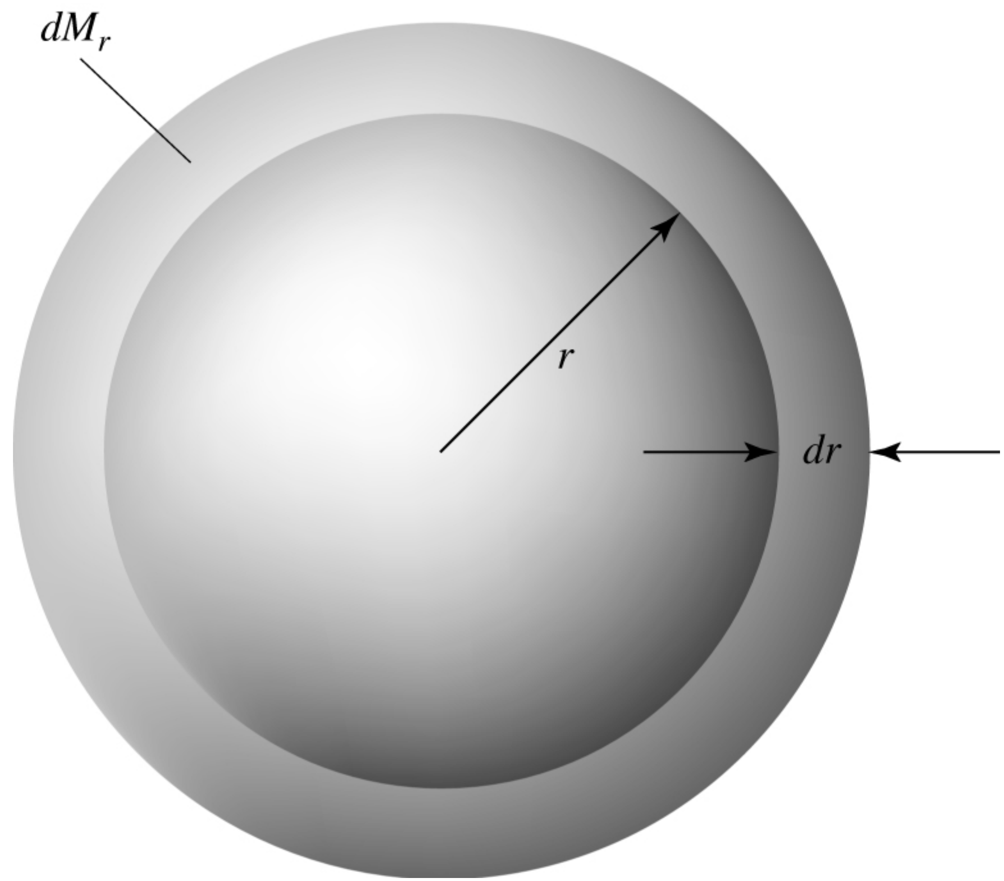
Conservation of Mass

The mass of a star must increase uniformly from the interior to the surface, there cannot be any holes! For each element of thickness *dr* the volume must increase by the mass of the shell encompassed, *i.e.* by the spherical area × thickness *dr*. The resulting equation is:

$$dM(r) = 4\pi r^2 \rho \, dr$$

or

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho$$



Ideal Gas Law

The ideal gas law is given by:

$$PV = NkT,$$

where P is the gas pressure, V its volume, N the number of particles in the volume, T is the temperature (on the absolute scale), and $k = 1.3807 \times 10^{-16}$ ergs/K is the Boltzmann constant.

It is also expressed using the gas constant R as:

$PV = nRT$, where n is the number of moles of gas.

The relationship can be derived from first principles through the kinetic theory of gases, which specifies that:

(i) gas consists of small particles (molecules, atoms) that are smaller than the distances separating them,
(ii) particles are in constant motion and make perfectly elastic collisions with container walls,
(iii) the motions of particles are random, *i.e.* ⅓ are moving in any specific direction.

Pressure Equation of State

The number density for the gas can be written as $n = N/V$, so:

$$P = nkT,$$

and since the number density can also be written as $n = \rho/\mu m_H$, where ρ is the actual density (gm/cm^3), m_H is the mass of a hydrogen atom, and μ is the mean molecular weight. So the gas pressure can be written as:

$$P_g = nkT = \frac{\rho kT}{\mu m_H}$$

Mean Molecular Weight

As the name implies, the mean molecular weight is the average mass of a gas particle, but in units of the mass of a hydrogen atom, *i.e.*:

$$\mu \equiv \frac{\overline{m}}{m_H}$$

where $\overline{m}$ **is the average mass of a gas particle.**

Consider possible examples:

Hydrogen: neutral,

$$\mu \equiv \frac{\overline{m}}{m_H} = \frac{m_H}{m_H} = 1$$

ionized,

$$\mu \equiv \frac{\overline{m}}{m_H} = \frac{\frac{1}{2} m_H}{m_H} = \frac{1}{2}$$

molecular,

$$\mu \equiv \frac{\overline{m}}{m_H} = \frac{2m_H}{m_H} = 2$$

Helium: neutral,

$$\mu \equiv \frac{\overline{m}}{m_H} = \frac{4m_H}{m_H} = 4$$

ionized,

$$\mu \equiv \frac{\overline{m}}{m_H} = \frac{\frac{4}{3}m_H}{m_H} = \frac{4}{3}$$

Heavy element: neutral,

$$\mu \equiv \frac{\overline{m}}{m_H} = \frac{2A_j m_H}{m_H} = 2A_j$$

ionized,

$$\mu \equiv \frac{\overline{m}}{m_H} \approx \frac{2A_j m_H}{\left(A_j + 1\right)m_H} \approx 2$$

where A_j is the atomic number.

In stellar interiors the value of μ for a *fully-ionized* gas is desired. How does one take into consideration the contributions from all elements?

Let X = fractional abundance by mass of hydrogen,
Y = fractional abundance by mass of helium, and
Z = fractional abundance by mass of all heavy elements,
where $X + Y + Z \equiv 1$, by definition.

In a cubic centimeter of gas of density ρ there is $X\rho$ of hydrogen, $Y\rho$ of helium, and $Z\rho$ of heavy elements by mass. Each of the elements contributes different numbers of electrons to the mix, under the assumption of complete ionization: 1 for hydrogen, 2 for helium, and 3, 4, 5… for the heavy elements. The number of particles per cubic centimeter is:

$X\rho/m_{\mathrm{H}} \times 2 = 2X\rho/m_{\mathrm{H}}$ for hydrogen

$Y\rho/4m_{\mathrm{H}} \times 3 = 3Y\rho/4m_{\mathrm{H}}$ for helium, and

$Z\rho/2A_j m_{\mathrm{H}} \times (A_j + 1) \approx Z\rho/2m_{\mathrm{H}}$ for the heavy elements.

Stellar Energy Sources

The gravitational potential energy required to contract a star to its present size is given by:

$$V = -\frac{3}{5}\frac{GM^2}{R}$$

But, of the potential energy lost by a star, according to the Virial Theorem, one half is transformed into an increase in the kinetic energy of the gas (heat) and the remainder is radiated into space. The radiation lost by a star upon contraction to the main sequence is therefore given by:

$$E = +\frac{3}{10}\frac{GM^2}{R}$$

For the Sun, at its present mass (1.9891 × 10^{33} gm) and radius (6.9598 × 10^{10} cm), the amount of energy radiated through contraction is:

$$E = +\frac{3}{10}\frac{\left(6.672\times10^{-8}\right)\left(1.9891\times10^{33}\right)^2}{\left(6.9598\times10^{10}\right)} = 1.138\times10^{48}\ \text{ergs}$$

The present luminosity of the Sun is L = 3.851 × 10^{33} ergs/s, so if it had been shining at the same luminosity for the entire duration of its contraction (clearly erroneous) then the time scale for contraction is given by the following:

$$\text{Energy Lost} = \text{Present Luminosity} \times \text{Time Scale for Contraction}$$

$$\text{or} \quad \Delta E = L_{\text{Sun}} \times t_{\text{KH}}$$

where t_{KH} is the Kelvin-Helmholtz time scale. Here:

$$t_{\text{KH}}(\text{Sun}) = \frac{\Delta E}{L_{\text{Sun}}} = \frac{1.138\times10^{48}\ \text{ergs}}{3.851\times10^{33}\ \text{ergs/s}} = \frac{2.955\times10^{14}\ \text{s}}{3.1557\times10^{7}\ \text{s/yr}} = 9.4\times10^{6}\ \text{yr}$$

or ~10^7 years. The actual value should be smaller because the Sun's luminosity was greater during the contraction phase, but the main point is that the estimate is much shorter than the estimated age of the solar system of ~4.6 × 10^9 years.

Nuclear Energy Sources

Consider the rest masses of the fundamental nuclear particles:

Proton: 1.672623×10^{-24} gm
Neutron: 1.674929×10^{-24} gm
Electron: 9.109390×10^{-28} gm

Atomic mass unit, 1 *u* = 1.660540×10^{-24} gm = 931.49432 MeV, for $E = mc^2$.

The original nucleon symbolism was: $^{A}_{Z}X$

where A = mass number = number of nucleons
Z = number of protons (usually omitted)
X = chemical symbol of the element as specified by Z.

***i.e.*, $^{1}_{1}H$ is redundant, since ^{1}H indicates the same thing.**

Typical masses:

^{1}H = 1.007825 *u* = 938.78326 MeV

^{2}H = 2.014102 *u* = 1876.12457 MeV

^{4}He = 4.002603 *u* = 3728.40196 MeV

^{5}Li = 5.0125 *u* = 4669.115279 MeV

^{8}Be = 8.005305 *u* = 7456.89614 MeV

The major reaction in astronomy converts 4 hydrogen nuclei (protons) into a helium nucleus (^{4}He).

But 4 ^{1}H = 1.007825 *u* × 4 = 4.031280 *u*

and 1 ^{4}He = 4.002603 *u* = 4.002603 *u*

Difference = 0.028677 *u* = 0.0071 of 4 ^{1}H

The energy released = mc^2 = 0.028677 *u* × 1.660540 × 10^{-24} gm c^2 = 26.71 MeV.

The lifetime of a star depends upon how much hydrogen content is converted to energy via nuclear reactions.

For the Sun we can estimate:

$$E_{\text{nuclear}} = 0.10 \times 0.0071 \times M_{\text{Sun}}\, c^2 = 1.3 \times 10^{51} \text{ ergs}$$

At $L = 3.851 \times 10^{33}$ ergs/s,

$$t_{\text{nuclear}}(\text{Sun}) = \frac{E_{\text{nuclear}}}{L_{\text{Sun}}} = \frac{1.3 \times 10^{51} \text{ ergs}}{3.851 \times 10^{33} \text{ ergs/s}} = \frac{3.38 \times 10^{17} \text{ s}}{3.1557 \times 10^{7} \text{ s/yr}} \cong 10^{10} \text{ yr}$$

1 M, $t_{\text{nuclear}} = 10^{10}$ years   2 M, $t_{\text{nuclear}} = 10^{9}$ years (A-star)
4.6 M, $t_{\text{nuclear}} = 10^{8}$ years   10 M, $t_{\text{nuclear}} = 10^{7}$ years (B-star)
21.5 M, $t_{\text{nuclear}} = 10^{6}$ years (O-star)
0.5 M, $t_{\text{nuclear}} = 10^{11}$ years > $1/H_0$ (estimated age of the universe)

The lifetime of the Sun and stars via nuclear reactions is consistent with the nuclear ages of meteorites, as well as the oldest rocks on the Earth and the Moon.